Reply to "Comment on 'Two-finger selection theory in the Saffman-Taylor problem'"

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We clarify the meaning of the results of Phys. Rev. E **60**, R5013 (1999). We discuss the use and implications of periodic boundary conditions, as opposed to rigid-wall ones. We briefly argue that the solutions of the paper above are physically relevant as part of a more general issue, namely the possible generalization to dynamics, of the microscopic solvability scenario of selection.

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In Ref. [1] we addressed the generalization of the well known single-finger selection phenomenon in the Saffman–Taylor problem, to stationary configurations where fingers with tips at different relative positions in the propagation direction and different widths could coexist. A continuum of such solutions was known to exist for zero surface tension, but it was not known whether a similar selection principle to that of the single-finger case would "quantize" them into a discrete set, or even if stationary solutions with unequal fingers existed at all for nonzero surface tension. In Ref. [2] it was also pointed out that this issue had not only an academic interest, but also had implications on the understanding of the generic mechanisms of Laplacian screening which are responsible for the dynamical process of finger competition. The central result of Ref. [1] was that surface tension does indeed select multifinger configurations. This result is not questioned by the Comment of Vasconcelos [3], who objects, though, to a possibly confusing use of periodic boundary conditions.

Although we obviously agree with the assertion that the use of rigid-wall boundary conditions would be closer to an experimental realization of this specific physical system, we would like to justify the use of periodic boundary conditions on the following theoretical grounds. The basic idea is that, from a general perspective in the context of interfacial pattern formation, one is interested in searching for generic dynamical mechanisms which underlie the phenomenon of finger competition, but which could be relevant to other related problems. It is therefore interesting to avoid as much as possible nongeneric details such as boundary effects. A common strategy in statistical and nonlinear physics is thus the use of periodic boundary conditions as a way to "soften" the boundaries. In the specific context of the Saffman-Taylor problem, periodic boundary conditions in the above sense are usually assumed (see for instance the review paper by Bensimon *et al.* [4]). Within this spirit, in Ref. [2] we proposed the class of time-dependent axisymmetric-finger solutions with periodic boundary conditions as the simplest subclass of solutions relevant to the general phenomenon of finger competition. By construction, those solutions were describing an infinite array of fingers with only two different fingers repeated alternatively, so we did refer to them as "two-finger" solutions, to emphasize that only two tip positions and two finger widths were considered. The study of such a configuration was proposed to capture the basic elementary process of finger competition.

In his Comment to Ref. [1], Vasconcelos [3] points out that the solutions studied there do not correspond to two fingers in a channel with rigid-wall boundary conditions, but to the rather artificial configuration of one finger flanked by two half fingers. We agree that the latter is indeed the configuration in which the rigid-wall boundary conditions are satisfied at the sidewalls. By assuming periodic boundary conditions, however, the location of the boundary of the unit cell which is periodically repeated turns out to be arbitrary, owing to the continuous translation invariance of the problem in an infinite channel. Similarly, Vasconcelos [3] claims that in Ref. [1] we are implicitly assuming a certain position of the branch cut of the mapping, which is only consistent with the configuration of his Fig. 1. In our formulation, which follows that of Ref. [4], the function $f(\omega)$ which maps the unit disk in the reference ω plane into the unit channel in the z plane, takes the form $f(\omega)=-\log\omega+h(\omega)$. We then enforce analyticity of $h(\omega)$ within the whole unit disk (which is actually a stronger condition than strict periodicity), while no specific boundary conditions must be satisfied at the two sides of the branch cut, as is the case for rigidwall boundary conditions (which break the translation invariance). In our case, instead, the location of the logarithmic branch cut is a matter of convention, and is irrelevant to the dynamics of the problem. As a matter of fact, with periodic boundary conditions in the above sense, a rotation in the ω plane, $\omega'=\exp(i\omega)$, must be a symmetry of the dynamics, so $\tilde{f}(\omega')=-\log\omega'+h(\omega')$ must correspond to the same (infinite) interface configuration, and the same time evolution, up to translations in the z plane. Notice that the infinite Riemann-sheet structure of the logarithm accommodates very naturally the mapping of the periodic replication of the unit strip.

We admit that the term "two-finger" used in Ref. [1], although natural in the context set by the prior discussion of Ref. [2], might be misleading to some extent, and that, strictly speaking, if one is interested in two-finger configurations in a (physically realizable) channel with rigid sidewalls, one should address the solvability analysis of the solutions proposed by Vasconcelos in his Comment. Unfortunately, this problem would have the additional difficulty of dealing with four selection parameters instead of two, which makes the analysis much more involved. As proposed by Vasconcelos, it would be interesting to know whether his four-parameter family of solutions yields unequal two-finger solutions in the limit of vanishing surface tension. There are indications, however, that this might not be the case. In fact, Tanveer [5] performed the solvability analysis of a two-parameter family of (nonaxisymmetric) single-finger solutions in a channel (with rigid walls), and found that only the axisymmetric subclass survived when surface tension was included. This leads one to presume that a similar phenomenon may occur to the solutions of Vasconcelos, and that only the ones studied in Ref. [1] would survive selection.

As a concluding remark, we would like to emphasize that, from our viewpoint, apart from the obvious argument of simplicity with respect to the four-parameter family of Vasconcelos, the theoretical relevance of our result of Ref. [1] relies on its implications on a dynamical systems approach to the Saffman-Taylor problem. This perspective was introduced in Ref. [2] and has been reviewed in Ref. [6], in an attempt to find a possible generalization of the microscopic solvability scenario of selection to the dynamics of the problem. According to this point of view, the selective role of surface tension is seen as a drastic modification of the structure of the phase space flow, which is signaled by the changes in the actual fixed points of the dynamics, when surface tension is introduced. In this way, the knowledge of the fixed points (stationary solutions) and their relative stability is essential to capture the global structure of the phase space flow [6]. Whether a true selection principle for the dynamics can be drawn from these insights in a sense similar to the scenario of selection of the single-finger case, however, still remains an open question which could be relevant to a broad class of interfacial pattern formation problems.

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